cult to treat. Oniashvili's book,[6] which makes some attack on problems of dynamically loaded shells, should be mentioned here, although that author's main interest is in the loading through pulsating membrane forces that would be encountered in earthquake resistance studies applied to building structures.

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Approximate Analytic Solutions for the Range of a Nonlifting Re-Entry Trajectory

THEODORE R. KORNREICH*

Polytechnic Institute of Brooklyn, Freeport, N. Y.

Nomenclature

A	= frontal area of re-entry body, ft^2
B_1,B_2,B_3,B_4	= functions described by Eqs. (17) and (18)
C_1,C_2	= constants in Eq. (18)
C_D	= drag coefficient
C_L	= lift coefficient
g	= gravitational acceleration, ft/sec^2
h	= altitude, ft
m	= mass, slugs
r	= distance from center of earth to re-entry body, ft
t	= time, sec
V	= velocity, fps
x	= range, ft
β	= constant in expression for exponential atmosphere 1 ft^{-1}/22,000
θ	= flight path angle, rad
ρ	= atmospheric density, slugs/ft^3

Subscripts

c	= circular, critical
e	= entry into earth's atmosphere
f	= condition when $\theta = 90°$
m	= minimum
s	= sea level

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* Research Associate, Aerodynamics Laboratory. Member AIAA.

THIS note presents the development of approximate analytic solutions for the evaluation of the range of re-entry trajectories. It is known that, if a body re-enters the atmosphere at subcircular velocity, its trajectory will be of the direct impact type, whereas if the re-entry velocity is supercircular, either a skip or direct impact trajectory will prevail, depending upon the values of re-entry angle and velocity. Therefore, as a prelude to the analysis of the range an investigation was carried out to establish a simple criterion for the determination of whether a given supercircular re-entry trajectory is of the skip or direct impact type.

Wang and Ting[1] presented the following expression applicable to skip trajectories, relating velocity, density, and entry angle:

$$\theta = \left[\theta_e^2 - \frac{C_L A}{m\beta}(\rho - \rho_e) - \frac{2}{\beta}\left(\frac{1}{r} - \frac{g}{V_e^2}\right)\ln\frac{\rho}{\rho_e}\right]^{1/2} \quad (1)$$

where subscript e denotes conditions at entry into the earth's atmosphere. Assuming $C_L = 0$ and evaluating (1) when entry angle reaches its minimum ($\theta = 0$),

$$\ln\frac{\rho_0}{\rho_e} = \frac{\beta\theta_e^2}{2g[(1/V_c^2) - (1/V_e^2)]} \quad (2)$$

where V_c is the orbital velocity at the earth's surface, equal to $(gr)^{1/2}$.

Table 1 Critical angle comparison for various re-entry conditions

V_e/V_c	m/C_DA, slugs/ft^2	θ_c, rad (analytic approx.)	θ_c, rad (machine calculation)
1.2	1.94	0.076	0.073
1.2	0.40	0.071	0.068
1.2	10.0	0.090	0.084
1.37[a]	1.94	0.153	0.149
1.4	1.94	0.108	0.099
1.7	1.94	0.128	0.119
1.7[a]	1.94	0.185	0.180

[a] These correspond to atmospheric entry altitude of 660,000 ft. Other cases were evaluated at 400,000 ft.

In the case of direct impact re-entry trajectories, the relationship of Allen and Eggers[2] may be used to calculate the density ρ^* corresponding to a prescribed set of re-entry conditions:

$$\rho^* - \rho_e = \frac{2m\beta \sin\theta_e}{C_DA}\ln\frac{V_e}{V} \quad (3)$$

For a prescribed supercircular re-entry velocity, it is of interest to determine the critical entry angle at which a trajectory switches from the direct impact to the skip type. When $V_e > V_c$, the value of θ_e decreases, reaches a minimum (corresponding to $V = V_c$), as seen from the equation of motion for $d\theta/dt$, and then increases once again. Therefore, a comparison of ρ^* and ρ_0 will indicate the type of trajectory to be expected for any given case, i.e., $\rho^* > \rho_0$ indicates skip trajectory; $\rho_0 > \rho^*$ indicates direct impact trajectory; and $\rho_0 = \rho^*$ specifies the critical entry angle.

The following iterative procedure may be used to determine θ_c: assume a value for θ_e and calculate ρ^* from Eq. (3). Use this value of ρ^* as a trial value for ρ_0 and calculate the corresponding θ_e from Eq. (2). If the assumed and calculated values of θ_e do not agree within tolerable limits, use the calculated value as the new trial value for θ_e and iterate once more. A rapid convergence toward the correct value will be obtained since the value of ρ^* varies little within a reasonably wide spread of values of entry angle.

The value of entry angle finally obtained, θ_c, is that at which the crossover point between skip and direct impact trajectories is reached. For a given set of trajectory parameters, $\theta_e < \theta_c$ gives rise to skip trajectories and $\theta_e > \theta_c$ gives rise to direct impact trajectories.

A comparison of results obtained, using the forementioned technique with numerical results presented by Ferri and Ting,[3] and with machine calculations specifically carried out in conjunction with this investigation, showed good agreement over a range of velocities from 30,000 to 45,000 fps, and over a range of values of m/C_DA from 0.4 to 10.0 slugs/ft^2. A comparison of numerical and analytical results is shown in Table 1.

The expression for range is given by

$$x = \int V \cos\theta dt \tag{4}$$

However, the greatest portion of the range occurs in the region where θ is near its minimum value so that series expansions can be made about θ_m. From the exponential atmosphere expression

$$\rho = \rho_s \exp(-\beta h) \tag{5}$$

and from the relation between velocity and altitude

$$dh/dt = -V \sin\theta \tag{6}$$

Eq. (4) reduces to

$$x = \frac{1}{\beta}\int_{\rho_e}^{\rho_f} \cot\theta \frac{d\rho}{\rho} \tag{7}$$

where subscript f denotes condition when flight path angle has reached 90°.

With $C_L = 0$,

$$\frac{d\theta}{d\rho} = -\frac{\cot\theta}{\rho\beta}\left(\frac{1}{r} - \frac{g}{V^2}\right) \tag{8}$$

For supercircular entry velocities, with $\theta_e > \theta_c$, expansion of θ about θ_m gives

$$\theta = \theta_m + \tfrac{1}{2}(\rho - \rho_m)^2(d^2\theta/d\rho^2)_m \tag{9}$$

since $(d\theta/d\rho)_m = 0$.

Using Eq. (3) in the form

$$\ln(V/V_e) = -(C_DA/2m\beta \sin\theta_e)(\rho - \rho_e) \tag{10}$$

the atmospheric density ρ_m corresponding to the value θ_m can be obtained for specified conditions of entry velocity, vehicle drag characteristics, and entry angle.

Differentiating (8),

$$\frac{d^2\theta}{d\rho^2} = \frac{\cot\theta}{\rho^2\beta}\left(\frac{1}{r} - \frac{g}{V^2}\right) \tag{11}$$

Evaluating (11) at conditions where $\theta = \theta_m$, and substituting into (9), allows the calculation of the value of θ_m. Thereafter, the utilization of (9) and (11), together with the known terminal value of θ, i.e., 90°, yields the value of atmospheric density, ρ_f, corresponding to this angle.

Equation (7) can be written as

$$x = \frac{1}{\beta}\int_{\rho_e}^{\rho_f}\left[\cot\theta_m + \frac{1}{2}\left\{\frac{d^2}{d\rho^2}(\cot\theta)\right\}_m(\rho - \rho_m)^2\right]\frac{d\rho}{\rho} \tag{12}$$

$$x = \frac{\cot\theta_m}{\beta}\ln\frac{\rho_f}{\rho_e} + \text{higher order terms} \tag{13}$$

Equation (13) was employed to evaluate the range for comparison with the numerical data presented in Ref. 3. It was found that the first term of (13) was approximately two orders of magnitude larger than the remainder of that equation. Comparison of numerical results indicated the approximate equation to be valid to within 5%. For example, using $V_e/V_c = 1.2$, $m/C_DA = 1.94$ slugs/ft^2, and $\theta_e = 0.11$ rad, the analytically determined range is 3.28×10^6 ft, whereas numerical evaluation yields 3.39×10^6 ft.

For subcircular entry velocities, θ will be a minimum at entry into the earth's atmosphere. Therefore,

$$\cot\theta = \cot\theta_e + [(d/d\rho)(\cot\theta)]_e(\rho - \rho_e) \tag{14}$$

Using Eq. (7), the expression for the range becomes

$$x = \frac{\cot\theta_e}{\beta}\ln\frac{\rho_f}{\rho_e} + \text{higher order terms} \tag{15}$$

The expansion scheme shown below, which was proposed by Wang and Ting,[4] can be used to obtain the value of ρ_f/ρ_e:

$$\cos\theta_f = \cos\theta_e + B_1(\rho_f - \rho_e) + B_2\ln(\rho_f/\rho_e) + B_3f_1(\rho) \tag{16}$$

where

$$B_1 = C_LA/2m\beta = 0 \qquad B_2 = \cos\theta_e/\beta r$$
$$B_3 = -(g\cos\theta_e/\beta V_e^2) \qquad B_4 = C_DA\rho_e/2m\beta\sin\theta_e \tag{17}$$

$$f_1(\rho) = \left[(1 + B_4C_1 + B_4^2C_2)\ln\frac{\rho_f}{\rho_e} - (B_4C_1 + 2B_4^2C_2)\frac{\rho_f - \rho_e}{\rho_e} + \frac{1}{2}B_4^2C_2\frac{(\rho_f^2 - \rho_e^2)}{\rho_e^2}\right] \tag{18}$$

The constant coefficients C_1 and C_2 are found from a collocation method in accordance with the spread of velocity change required, using the expansion

$$\frac{g}{V^2} = \frac{g}{V_e^2}\left[1 + C_1\ln\frac{V}{V_e} + C_2\left(\ln\frac{V}{V_e}\right)^2\right]$$

Once more, as in the case of supercircular entry velocity, almost the entire contribution to the range is provided by the first term of the governing range expression, which is Eq. (15) in this case.

Comparison of results, using Eq. (15) and machine calculations for entry angles in an interval from 3° to 45°, showed fairly good agreement with substantially less error incurred as the entry angle is increased. For example, with $m/C_DA = 1.94$ slugs/ft^2, $\theta_e = 3°$, and $V_e/V_c = 0.95$, the analytically determined range was 6.20×10^6 ft, whereas the value obtained numerically for the range was 4.70×10^6 ft; with $\theta_e = 45°$ and other parameters remaining the same, the analytical range was 3.90×10^5 ft, whereas machine calculation gave 3.56×10^5 ft. This variation in error can be attributed to the more precise applicability of Eq. (10), at comparatively large entry angles. Note that the analytical approximation presented yields results closer to the correct values than use of Kepler's Law. For $\theta_e = 3°$, the latter gives a value for the range of 7.75×10^6 ft.

Approximate solutions for the range and entry angle criterion for lifting re-entry trajectories will be reported in the near future.

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